On the Expansion of $\phi(x+h)$.

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The object of this paper is the development of Taylor's formula, the development of the form of the functional coefficient of h (usually known as θ), in the remainder in that formula, and the development of the forms of θ_1, $\theta_2 \ldots$ as they appear in the equations $(a)'$, $(b)' \ldots$

Suppose ϕx and $\phi' x$ to be finite and continuous functions for all values of x from $x = x'$ to $x = x' + mh$, m being always positive, and either a constant, or a function of x and h of such form as to reduce to a constant when $h = 0$. Then will

$$\frac{\phi(x' + mh) - \phi x'}{mh} = \phi'(x' + \theta' h),$$

where θ' is between 0 and m.

Let A and B be the algebraically greatest and least values of $\phi' x$ for values of x between x' and $x' + mh$.

Put

$$y = Ax - \phi x, \tag{1}$$

and

$$z = \phi x - Bx. \tag{2}$$

Then

$$\frac{dy}{dx} = A - \phi' x;$$

and

$$\frac{dz}{dx} = \phi' x - B,$$

both of which are positive for all values of x between x' and $x' + mh$, and consequently y and z are increasing functions for all values of x between x' and $x' + mh$.

Let x take each of the two values x' and $x'+mh$ in both (1) and (2). Then

$$y'' = A(x'+mh) - \phi(x'+mh),$$

$$z'' = \phi(x'+mh) - B(x+mh),$$

$$y' = Ax' - \phi x',$$

and

$$z' = \phi x - Bx'.$$

Then

$$\frac{y''-y'}{mh} = A - \frac{\phi(x'+mh)-\phi x'}{mh}, \quad \text{and} \quad \frac{z''-z'}{mh} = \frac{\phi(x'+mh)-\phi x'}{mh} - B.$$

Because y and z are increasing functions, and m is positive, $y''-y'$ and $z''-z'$ are of the same sign as h, and the members of the last two equations are positive. Consequently $\frac{\phi(x'+mh)-\phi x'}{mh}$ is less than A and greater than B, and therefore, by reason of the continuity of $\phi' x$ between A and B, is equal to some value of $\phi' x$ between A and B in which x has some value between x' and $x'+mh$. Let $x'+\theta' h$ represent this value of x, θ' being between 0 and m. Then

$$\frac{\phi(x'+mh)-\phi x'}{mh} = \phi'(x'+\theta' h),$$

or

$$\phi(x+h) = \phi x + mh\phi'(x+\theta' h), \tag{3}$$

which will hold true for all values of x and mh within the limits of those giving finite and continuous values in ϕx and $\phi' x$.

Differentiating (3) twice, regarding x as constant, and in the result putting $h=0$,

$$(m^2)_{h=0}\phi'' x = 2(m\theta')_{h=0}\phi'' x.$$

$$\therefore\ (\theta')_{h=0} = \frac{1}{2}(m)_{h=0}, \quad \text{and} \quad \theta' = \frac{1}{2}(m)_{h=0} + \theta^0,$$

where $\theta^0 = 0$ when $h=0$. These results obtain whether m be constant or variable.

Since θ' is always between 0 and m, and m is always positive, and since θ' reduces to the constant $\frac{1}{2}(m)_{h=0}$ when $h=0$, θ', like m, is always positive, and either a constant, or reduces to one when $h=0$. Hence, if $\phi'' x$, $\phi''' x \ldots\ldots \phi^{2n} x$ are finite and continuous functions the same as ϕx and $\phi' x$, we can form the

following equations, in which θ'', θ''' .... are, like m and θ', always positive, and either constants, or reduced to constants when $h=0$.

$$\phi(x+mh)=\phi x+mh\phi'(x+\theta' h), \qquad (a)$$

where θ' is between 0 and m;

$$\phi'(x+\theta' h)=\phi' x+\theta' h\phi''(x+\theta'' h), \qquad (b)$$

where θ'' is between 0 and θ';

$$\phi''(x+\theta'' h)=\phi'' x+\theta'' h\phi'''(x+\theta''' h), \qquad (c)$$

where θ''' is between 0 and θ'';

. . . .

$$\phi^{2n-1}(x+\theta^{2n-1}h)=\phi^{2n-1}x+\theta^{2n-1}h\phi^{2n}(x+\theta^{2n}h), \qquad (d)$$

where θ^{2n} is between 0 and θ^{2n-1}.

The same law that makes

$$(\theta')_{h=0}=\frac{1}{2}(m)_{h=0},$$

also makes

$$(\theta'')_{h=0}=\frac{1}{2}(\theta')_{h=0}=\frac{1}{2^2}(m)_{h=0},$$

$$(\theta''')_{h=0}=\frac{1}{2}(\theta'')_{h=0}=\frac{1}{2^3}(m)_{h=0},$$

and in general,

$$(\theta^{2n})_{h=0}=\frac{1}{2^{2n}}(m)_{h=0}.$$

Let $m=1$, and denote what θ', θ'', θ'''.... become by θ_1, θ_2, θ_3 .... Then,

$$\phi(x+h)=\phi x+h\phi'(x+\theta_1 h), \qquad (a)$$

where θ_1 is between 0 and 1;

$$\phi'(x+\theta_1 h)=\phi' x+\theta_1 h\phi''(x+\theta_2 h), \qquad (b)$$

where θ_2 is between 0 and θ_1;

$$\phi''(x+\theta_2 h)=\phi'' x+\theta_2 h\phi''' x(x+\theta_3 h), \qquad (c)$$

where θ_3 is between 0 and θ_2;

. . . .

$$\phi^{2n-1}(x+\theta_{2n-1}h)=\phi^{2n-1}x+\theta_{2n-1}h\phi^{2n}(x+\theta_{2n}h), \qquad (d)$$

where θ_{2n} is between 0 and θ_{2n-1}.

Further, if $h=0$, then

$$(\theta_1)_{h=0}=\frac{1}{2}, \qquad (\theta_2)_{h=0}=\frac{1}{4},$$

$$(\theta_3)_{h=0}=\frac{1}{8}\cdots\cdot(\theta_{2n})_{h=0}=\frac{1}{2^{2n}}.$$

Forming the successive diff. co. of (3), regarding x as constant, and in the results putting $h=0$,

$$(m)_{h=0}\phi'x=(m)_{h=0}\phi'x. \tag{4}$$

$$(m^2)_{h=0}\phi''x=2(m\theta')_{h=0}\phi''x. \tag{5}$$

$$6\left(m\frac{dm}{dh}\right)_{h=0}\phi''x+(m^3)_{h=0}\phi'''x=6\left(\theta'\frac{dm}{dh}\right)_{h=0}\phi''x+6\left(m\frac{d\theta'}{dh}\right)_{h=0}\phi''x+3(m\theta'^2)_{h=0}\phi'''x. \tag{6}$$

$$\begin{aligned}12\left(m\frac{d^2m}{dh^2}\right)_{h=0}\phi''x+12\left(\frac{dm}{dh}\right)^2_{h=0}\phi''x+12\left(m^2\frac{dm}{dh}\right)_{h=0}\phi'''x+(m^4)_{h=0}\phi^{\text{IV}}x=12\left(\theta'\frac{d^2m}{dh^2}\right)_{h=0}\phi''x\\+24\left(\frac{dm}{dh}\frac{d\theta'}{dh}\right)_{h=0}\phi''x+12\left(\theta'^2\frac{dm}{dh}\right)_{h=0}\phi'''x+12\left(m\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi''x+24\left(m\theta'\frac{d\theta'}{dh}\right)_{h=0}\phi'''x\\+4(m\theta'^3)_{h=0}\phi^{\text{IV}}x.\end{aligned} \tag{7}$$

$$\begin{aligned}20\left(m\frac{d^3m}{dh^3}\right)_{h=0}\phi''x+60\left(\frac{dm}{dh}\frac{d^2m}{dh^2}\right)_{h=0}\phi''x+30\left(m^2\frac{d^2m}{dh^2}\right)_{h=0}\phi'''x+60\left[m\left(\frac{dm}{dh}\right)^2\right]_{h=0}\phi'''x\\+20\left(m^3\frac{dm}{dh}\right)_{h=0}\phi^{\text{IV}}x+(m^5)_{h=0}\phi^{\text{V}}x=20\left(\theta'\frac{d^3m}{dh^3}\right)_{h=0}\phi''x+60\left(\frac{d^2m}{dh^2}\frac{d\theta'}{dh}\right)_{h=0}\phi''x\\+30\left(\theta'^2\frac{d^2m}{dh^2}\right)_{h=0}\phi'''x+60\left[m\left(\frac{d\theta'}{dh}\right)^2\right]_{h=0}\phi'''x+20\left(\theta'^3\frac{dm}{dh}\right)_{h=0}\phi^{\text{IV}}x+5(m\theta'^4)_{h=0}\phi^{\text{V}}x\\+60\left(\frac{dm}{dh}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi''x+120\left(\theta'\frac{dm}{dh}\frac{d\theta'}{dh}\right)_{h=0}\phi'''x+20\left(m\frac{d^3\theta'}{dh^3}\right)_{h=0}\phi''x+60\left(m\theta'\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'''x\\+60\left(m\theta'^2\frac{d\theta'}{dh}\right)_{h=0}\phi^{\text{IV}}x.\end{aligned} \tag{8}$$

$$\begin{aligned}30\left(m\frac{d^4m}{dh^4}\right)_{h=0}\phi''x+120\left(\frac{dm}{dh}\frac{d^3m}{dh^3}\right)_{h=0}\phi''x+60\left(m^2\frac{d^3m}{dh^3}\right)_{h=0}\phi'''x+90\left(\frac{d^2m}{dh^2}\right)^2_{h=0}\phi''x\\+360\left(m\frac{dm}{dh}\frac{d^2m}{dh^2}\right)_{h=0}\phi'''x+60\left(m^2\frac{d^2m}{dh^2}\right)_{h=0}\phi^{\text{IV}}x+120\left(\frac{dm}{dh}\right)^3_{h=0}\phi'''x\\+180\left[m^2\left(\frac{dm}{dh}\right)^2\right]_{h=0}\phi^{\text{IV}}x+30\left(m^4\frac{dm}{dh}\right)_{h=0}\phi^{\text{V}}x+(m^6)_{h=0}\phi^{\text{VI}}x=30\left(\theta'\frac{d^4m}{dh^4}\right)_{h=0}\phi''x\\+120\left(\frac{d^3m}{dh^3}\frac{d\theta'}{dh}\right)_{h=0}\phi''x+60\left(\theta'^2\frac{d^3m}{dh^3}\right)_{h=0}\phi'''x+180\left(\frac{d^2m}{dh^2}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi''x\end{aligned}$$

$$+ 360\left(\theta'\frac{d^2m}{dh^2}\frac{d\theta'}{dh}\right)_{h=0}\phi'''x + 60\left(\theta'^3\frac{d^2m}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}}x + 120\left(\frac{dm}{dh}\frac{d^3\theta'}{dh^3}\right)_{h=0}\phi''x$$

$$+ 360\left(\theta'\frac{dm}{dh}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'''x + 360\left[\frac{dm}{dh}\left(\frac{d\theta'}{dh}\right)^2\right]_{h=0}\phi'''x + 360\left(\theta'^2\frac{dm}{dh}\frac{d\theta'}{dh}\right)_{h=0}\phi^{\mathrm{IV}}x$$

$$+ 30\left(m\frac{d^4\theta'}{dh^4}\right)_{h=0}\phi''x + 120\left(m\theta'\frac{d^3\theta'}{dh^3}\right)_{h=0}\phi'''x + 360\left(m\frac{d\theta'}{dh}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'''x$$

$$+ 180\left(m\theta'^2\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}}x + 30\left(\theta'^4\frac{dm}{dh}\right)_{h=0}\phi^{\mathrm{V}}x + 360\left[m\theta'\left(\frac{d\theta'}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{IV}}x$$

$$+ 120\left(m\theta'^3\frac{d\theta'}{dh}\right)_{h=0}\phi^{\mathrm{V}}x + 6\,(m\theta'^5)_{h=0}\,\phi^{\mathrm{VI}}x. \tag{9}$$

. . . .

If in (4), (5), (6), (7), (8), and (9) we put $m = 1$, and write θ_1 for θ', we obtain the same results as when $h = 0$ in the successive diff. co., with respect to h, of $(a)'$. Hence,

$$\phi'x = \phi'x. \tag{10}$$

$$\phi''x = 2\,(\theta_1)_{h=0}\phi''x. \tag{11}$$

$$\phi'''x = 6\left(\frac{d\theta_1}{dh}\right)_{h=0}\phi''x + 3\,(\theta_1^2)_{h=0}\,\phi'''x. \tag{12}$$

$$\phi^{\mathrm{IV}}x = 12\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi''x + 24\left(\theta_1\frac{d\theta_1}{dh}\right)_{h=0}\phi'''x + 4\,(\theta_1^3)_{h=0}\phi^{\mathrm{IV}}x. \tag{13}$$

$$\phi^{\mathrm{V}}x = 20\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi''x + 60\left(\theta_1\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi'''x + 60\left(\frac{d\theta_1}{dh}\right)^2_{h=0}\phi'''x + 60\left(\theta_1^2\frac{d\theta_1}{dh}\right)_{h=0}\phi^{\mathrm{IV}}x$$
$$+ 5\,(\theta_1^4)_{h=0}\phi^{\mathrm{V}}x. \tag{14}$$

$$\phi^{\mathrm{VI}}x = 30\left(\frac{d^4\theta_1}{dh^4}\right)_{h=0}\phi''x + 120\left(\theta_1\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi'''x + 360\left(\frac{d\theta_1}{dh}\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi'''x + 180\left(\theta_1^2\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}}x$$
$$+ 360\left[\theta_1\left(\frac{d\theta_1}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{IV}}x + 120\left(\theta_1^3\frac{d\theta_1}{dh}\right)_{h=0}\phi^{\mathrm{V}}x + 6\,(\theta_1^5)_{h=0}\phi^{\mathrm{VI}}x. \tag{15}$$

Similarly,

$$\phi^{\mathrm{VII}}x = 42\left(\frac{d^5\theta_1}{dh^5}\right)_{h=0}\phi''x + 210\left(\theta_1\frac{d^4\theta_1}{dh^4}\right)_{h=0}\phi'''x + 840\left(\frac{d\theta_1}{dh}\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi'''x + 420\left(\theta_1^2\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi^{\mathrm{IV}}x$$
$$+ 630\left(\frac{d^2\theta_1}{dh^2}\right)^2_{h=0}\phi'''x + 2520\left(\theta_1\frac{d\theta_1}{dh}\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}}x + 840\left(\frac{d\theta_1}{dh}\right)^3\phi^{\mathrm{IV}}x$$
$$+ 1260\left[\theta_1^2\left(\frac{d\theta_1}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{V}}x + 420\left(\theta_1^3\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi^{\mathrm{V}}x + 210\left(\theta_1^4\frac{d\theta_1}{dh}\right)_{h=0}\phi^{\mathrm{VI}}x$$
$$+ 7\,(\theta_1^6)_{h=0}\phi^{\mathrm{VII}}x. \tag{16}$$

. . . .

Equation $(a)'$ reduces to $\phi x = \phi x$, when $h = 0$, but its diff. co., with respect to h, reduces to $\phi' x = \phi' x$, by (10). Hence by one differentiation with respect to h, h has been eliminated from one term in $h\phi'(x + \theta_1 h)$. Consequently one term in $h\phi'(x + \theta_1 h)$ is $h\phi' x$.

Again: when $h = 0$, the second diff. co. of $(a)'$ with respect to h, reduces to $\phi'' x = 2(\theta_1)_{h=0}\phi'' x$, by (11). Consequently by two differentiations with respect to h, $\frac{h^2}{2}$ has been eliminated from a second term in $h\phi'(x + \theta_1 h)$, and a second term in that expression is

$$\frac{h^2}{2}\phi'' x = \frac{h^2}{2}\left[2(\theta_1)_{h=0}\phi'' x\right].$$

In like manner by three differentiations with respect to h, and in the result putting $h = 0$, (12) is obtained from $(a)'$, where it is plain $\frac{h^3}{2.3}$ has been eliminated from a third term in $h\phi'(x + \theta_1 h)$, and a third term in in that expression is

$$\frac{h^3}{2.3}\phi''' x = \frac{h^3}{2.3}\left[6\left(\frac{d\theta_1}{dh}\right)_{h=0}\phi'' x + 3(\theta_1^2)_{h=0}\phi''' x\right].$$

Additional terms may be found in the same way from (13), (14) . . . ., and with the foregoing ones substituted for $h\phi'(x + \theta_1 h)$ in $(a)'$, giving

$$\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h_3}{2.3}\phi''' x + \ldots . \frac{h^n}{\lfloor n}\phi^n x + \ldots . \tag{17}$$

If, in (17), $\frac{h^n}{\lfloor n}\phi^n(x + \theta h)$ denote the sum of all the terms in the right member after the first n, and again $\frac{h^m}{\lfloor m} R$ denote the sum of all the terms after the first m, m being any integer between n and $2n$, we can write the two equations

$$\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \ldots . \frac{h^{n-1}}{\lfloor n-1}\phi^{n-1} x + \frac{h^n}{\lfloor n}\phi^n(x+\theta h). \tag{18}$$

$$\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \ldots . \frac{h^n}{\lfloor n}\phi^n x + \frac{h^{n+1}}{\lfloor n+1}\phi^{n+1} x + \ldots . \frac{h^{m-1}}{\lfloor m-1}\phi^{m-1} x + \frac{h^m}{\lfloor m} R. \tag{19}$$

From (18) and (19),

$$\phi^n(x+\theta h) = \phi^n x + \frac{h}{(n+1)}\phi^{n+1} x + \frac{h^2}{(n+1)(n+2)}\phi^{n+2} x + \ldots . \frac{h^{m-n}}{(n+1)(n+2)\ldots . m} R. \tag{20}$$

By differentiating (18) n times, regarding x as constant,

$$\phi^n(x+h) = \phi^n(x+\theta h) + nh\frac{d\phi^n(x+\theta h)}{dh} + \frac{n(n-1)h^2}{2}\frac{d^2\phi^n(x+\theta h)}{dh^2} + \ldots . \frac{h^n}{\lfloor n}\frac{d^n\phi^n(x+\theta h)}{dh^n}. \tag{21}$$

From (18), when $n=0$,

$$\phi(x+h)=\frac{h^0}{1}\phi^0(x+\theta h)=\phi^0(x+\theta h), \quad \text{and} \quad \theta=1. \tag{22}$$

If $n=\infty$, then, by (20),

$$\phi^n(x+\theta h)=\phi^n x, \text{ and } \theta=0. \tag{23}$$

From (20) and (21) we find the quantity $\phi^n(x+\theta h)$ can never equal either $\phi^n(x+h)$, or $\phi^n x$, unless n has such a value as to cause all the terms after the first, in the right members of (20) and (21), to vanish.

This can only happen in (20) when n is infinite, and in (21) when n is zero. Hence for finite integral values of n, θ is a positive proper fraction, or θ is between zero and a unit.

Differentiating (20), regarding x as constant,

$$\theta\phi^{n+1}(x+\theta h)+\frac{d\theta}{dh}h\phi^{n+1}(x+\theta h)=\frac{1}{n+1}\phi^{n+1}x+\frac{2h}{(n+1)(n+2)}\phi^{n+2}x$$
$$+\frac{3h^2}{(n+1)(n+2)(n+3)}\phi^{n+3}x+\ldots\frac{(m-n)h^{m-n-1}}{(n+1)\ldots m}R'+\frac{h^{m-n}}{(n+1)(n+2)\ldots m}\frac{dR'}{dh}. \tag{24}$$

In (24), when $h=0$,

$$(\theta)_{h=0}\phi^{n+1}x=\frac{1}{n+1}\phi^{n+1}x, \quad \text{and} \quad (\theta)_{h=0}=\frac{1}{n+1}. \tag{25}$$

Continuing the differentiation of (24), regarding x as constant, and in the results putting $h=0$,

$$2\left(\frac{d\theta}{dh}\right)_{h=0}\phi^{n+1}x+(\theta^2)_{h=0}\phi^{n+2}x=\frac{2\phi^{n+2}x}{(n+1)(n+2)}. \tag{26}$$

$$3\left(\frac{d^2\theta}{dh^2}\right)_{h=0}\phi^{n+1}x+6\left(\theta\frac{d\theta}{dh}\right)_{h=0}\phi^{n+2}x+(\theta^3)_{h=0}\phi^{n+3}x=\frac{2.3\,\phi^{n+3}x}{(n+1)(n+2)(n+3)} \tag{27}$$

$$4\left(\frac{d^3\theta}{dh^3}\right)_{h=0}\phi^{n+1}x+12\left(\theta\frac{d^2\theta}{dh^2}\right)_{h=0}\phi^{n+2}x+12\left(\frac{d\theta}{dh}\right)^2_{h=0}\phi^{n+2}x+12\left(\theta^2\frac{d\theta}{dh}\right)_{h=0}\phi^{n+3}x$$
$$+(\theta^4)_{h=0}\phi^{n+4}x=\frac{2.3.4\,\phi^{n+4}x}{(n+1)(n+2)\ldots(n+4)}. \tag{28}$$

. . . .

From (25) and (26),

$$\left(\frac{d\theta}{dh}\right)_{h=0}=\left(\frac{2(n+1)-(n+2)}{2(n+1)^2.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right). \tag{29}$$

Again, from (27), by the aid of (25) and (29),

$$\left(\frac{d^2\theta}{dh^2}\right)_{h=0}=\left(\frac{2.3(n+1)^2-(n+2)(n+3)}{3(n+1)^3.(n+2)(n+3)}\right)\left(\frac{\phi^{n+3}x}{\phi^{n+1}x}\right)-\left(\frac{2(n+1)-(n+2)}{(n+1)^3.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)^2. \tag{30}$$

In like manner it may be shown, from (28), that

$$\left(\frac{d^3\theta}{dh^3}\right) = \left(\frac{2.3.4(n+1)^3 - (n+2)(n+3)(n+4)}{2^2(n+1)^4.(n+2)(n+3)(n+4)}\right)\left(\frac{\phi^{n+4}x}{\phi^{n+1}x}\right) - \left(\frac{2.3(n+1) - 3(n+2)}{2(n+1)^4.(n+2)}\right.$$
$$+ \left.\frac{2.3^2(n+1)^2 - 3(n+2)(n+3)}{3(n+1)^4.(n+2)(n+3)}\right)\left(\frac{\phi^{n+3}x}{\phi^{n+1}x}\cdot\frac{\phi^{n+2}x}{\phi^{n+1}x}\right) - \left(\frac{2^2.3(n+1)^2 - 2^2.3(n+1)(n+2) + 3(n+2)^2}{2^2(n+1)^4.(n+2)^2}\right.$$
$$\left. - \frac{2.3(n+1) - 3(n+2)}{(n+1)^4.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)^2. \tag{31}$$

In the same way, the values of $\left(\frac{d^4\theta}{dh^4}\right)_{h=0}$, $\left(\frac{d^5\theta}{dh^5}\right)_{h=0}$ .... may be found.

From (25), $\theta = \frac{1}{n+1} + \phi$, where $\phi = 0$ when $h = 0$; but by one differentiation with respect to h, and in the result putting $h = 0$,

$$\left(\frac{d\theta}{dh}\right)_{h=0} = \left(\frac{d\phi}{dh}\right)_{h=0} = \left(\frac{2(n+1) - (n+2)}{2(n+1)^2.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right),$$

by (29). Consequently, by one differentiation with respect to h, h has been eliminated from one term in ϕ, and also from a second term in θ. Hence, before differentiation,

$$\theta = \frac{1}{n+1} + h\left(\frac{d\theta}{dh}\right)_{h=0} + \phi' = \frac{1}{n+1} + h\left[\frac{2(n+1) - (n+2)}{2(n+1)^2.(n+2)}\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)\right] + \phi',$$

where ϕ' and $\left(\frac{d\phi'}{dh}\right)$ both vanish when $h = 0$. Differentiating the last equation twice, regarding x as constant, and in the result putting $h = 0$, $\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} = \left(\frac{d^2\phi'}{dh^2}\right)_{h=0}$ = the right member (30). Hence by two differentiations with respect to h, $\frac{h^2}{2}$ has been eliminated from a term in ϕ', and from a third term in θ. Consequently,

$$\theta = \frac{1}{n+1} + h\left(\frac{d\theta}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta}{dh^2}\right)_{h=0} + \phi'' = \frac{1}{n+1} + h\left[\left(\frac{2(n+1) - (n+2)}{2(n+1)^2.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)\right]$$
$$+ \frac{h^2}{2}\left[\left(\frac{2.3(n+1)^2 - (n+2)(n+3)}{3(n+1)^3.(n+2).(n+3)}\right)\left(\frac{\phi^{n+3}x}{\phi^{n+1}x}\right) - \left(\frac{2(n+1) - (n+2)}{(n+1)^3.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)^2\right] + \phi'',$$

where ϕ'', $\frac{d\phi''}{dh}$, and $\frac{d^2\phi''}{dh^2}$ vanish when $h = 0$. In like manner additional terms may be found, and the general form of θ shown to be,

$$\theta = \frac{1}{n+1} + h\left(\frac{d\theta}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta}{dh^3}\right)_{h=0} + \dots \frac{h^{n-1}}{\lfloor n-1}\left(\frac{d^{n-1}\theta}{dh^{n-1}}\right)_{h=0} + \dots, \tag{32}$$

where $\left(\frac{d\theta}{dh}\right)_{h=0}, \left(\frac{d^2\theta}{dh^2}\right)_{h=0}, \ldots.$ are expressible in terms $\phi^{n+1}x, \phi^{n+2}x, \ldots. \phi^{2n}x$, as in (29), (30), (31).

From (11), (12), ... (16), it may readily be shown that

$$\left(\frac{d\theta_1}{dh}\right)_{h=0} = \frac{1}{24}\frac{\phi'''x}{\phi''x}. \tag{33}$$

$$\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} = \frac{1}{24}\frac{\phi^{\text{IV}}x}{\phi''x} - \frac{1}{24}\left(\frac{\phi'''x}{\phi''x}\right)^2. \tag{34}$$

$$\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0} = \frac{11}{320}\frac{\phi^{\text{V}}x}{\phi''x} - \frac{3}{32}\frac{\phi^{\text{IV}}x}{\phi''x}\frac{\phi'''x}{\phi''x} + \frac{11}{192}\left(\frac{\phi'''x}{\phi''x}\right)^3. \tag{35}$$

$$\left(\frac{d^4\theta_1}{dh^4}\right)_{h=0} = \frac{13}{480}\frac{\phi^{\text{VI}}x}{\phi''x} - \frac{43}{480}\frac{\phi^{\text{V}}x}{\phi''x}\frac{\phi'''x}{\phi''x} + \frac{7}{32}\frac{\phi^{\text{IV}}x}{\phi''x}\left(\frac{\phi'''x}{\phi''x}\right)^2 - \frac{1}{16}\left(\frac{\phi^{\text{IV}}x}{\phi''x}\right)^2 - \frac{3}{32}\left(\frac{\phi'''x}{\phi''x}\right)^4, \tag{36}$$

$$\begin{aligned}\left(\frac{d^5\theta_1}{dh^5}\right)_{h=0} = {} & \frac{19}{896}\frac{\phi^{\text{VII}}x}{\phi''x} - \frac{31}{384}\frac{\phi^{\text{VI}}x}{\phi''x}\frac{\phi'''x}{\phi''x} + \frac{15}{64}\frac{\phi^{\text{V}}x}{\phi''x}\left(\frac{\phi'''x}{\phi''x}\right)^2 - \frac{55}{108}\frac{\phi^{\text{IV}}x}{\phi''x}\left(\frac{\phi'''x}{\phi''x}\right)^3 + \frac{5}{16}\left(\frac{\phi^{\text{IV}}x}{\phi''x}\right)^2\frac{\phi'''x}{\phi''x} \\ & - \frac{53}{384}\frac{\phi^{\text{V}}x}{\phi''x}\frac{\phi^{\text{IV}}x}{\phi''x} + \frac{185}{1152}\left(\frac{\phi'''x}{\phi''x}\right)^5.\end{aligned} \tag{37}$$

. . . .

The value of θ_1 may be found from those of

$$\left(\frac{d\theta_1}{dh}\right)_{h=0}, \left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}, \left(\frac{d^3\theta_1}{dh^3}\right)_{h=0}, \ldots.$$

in the same manner that the value of θ was derived from those of

$$\left(\frac{d\theta}{dh}\right)_{h=0}, \left(\frac{d^2\theta}{dh^2}\right)_{h=0}, \ldots.$$

Hence,

$$\theta_1 = \frac{1}{2} + h\left[\frac{1}{24}\frac{\phi'''x}{\phi''x}\right] + \frac{h^2}{2}\left[\frac{1}{24}\frac{\phi^{\text{IV}}x}{\phi''x} - \frac{1}{24}\left(\frac{\phi'''x}{\phi''x}\right)^2\right] + \ldots., \tag{38}$$

or,

$$\theta_1 = \frac{1}{2} + h\left(\frac{d\theta_1}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0} + \ldots. \frac{h^n}{\lfloor n}\left(\frac{d^n\theta_1}{dh^n}\right)_{h=0} + \ldots., \tag{39}$$

where

$$\left(\frac{d\theta_1}{dh}\right)_{h=0}, \left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}, \ldots. \left(\frac{d^n\theta_1}{dh^n}\right)_{h=0}$$

are expressible in terms of $\phi''x, \phi'''x, \ldots. \phi^{n+2}x$.

If in (5), (6), (7), (8) and (9) we write θ_1 for m, θ_2 for θ, and advance each order of diff. co. of x to the next higher, we obtain the same results as when

$h=0$ in the successive diff. co. with respect to h, of $(b)'$, and these results by easy reductions give

$$\left(\frac{d\theta_2}{dh}\right)_{h=0}=\frac{1}{96}\frac{\phi^{\mathrm{IV}}x}{\phi'''x}+\frac{1}{48}\frac{\phi'''x}{\phi''x}. \tag{40}$$

$$\left(\frac{d^2\theta_2}{dh^2}\right)_{h=0}=\frac{1}{192}\frac{\phi^{\mathrm{V}}x}{\phi'''x}-\frac{1}{192}\left(\frac{\phi^{\mathrm{IV}}x}{\phi'''x}\right)^2+\frac{7}{288}\frac{\phi^{\mathrm{IV}}x}{\phi''x}-\frac{1}{48}\left(\frac{\phi'''x}{\phi''x}\right)^2. \tag{41}$$

$$\left(\frac{d^3\theta_2}{dh^3}\right)_{h=0}=\frac{11}{5120}\frac{\phi^{\mathrm{VI}}x}{\phi'''x}-\frac{3}{512}\frac{\phi^{\mathrm{V}}x}{\phi'''x}\frac{\phi^{\mathrm{IV}}x}{\phi'''x}+\frac{11}{3072}\left(\frac{\phi^{\mathrm{IV}}x}{\phi'''x}\right)^2+\frac{27}{1280}\frac{\phi^{\mathrm{V}}x}{\phi''x}+\frac{1}{768}\frac{\phi^{\mathrm{IV}}x}{\phi''x}\frac{\phi^{\mathrm{IV}}x}{\phi'''x}$$
$$-\frac{119}{2304}\frac{\phi^{\mathrm{IV}}x}{\phi''x}\frac{\phi'''x}{\phi''x}+\frac{11}{384}\left(\frac{\phi'''x}{\phi''x}\right)^3. \tag{42}$$

. . . .

From these values of $\left(\frac{d\theta_2}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_2}{dh^2}\right)_{h=0}$, ..., we find

$$\theta_2=\frac{1}{4}+h\left[\frac{1}{96}\frac{\phi^{\mathrm{IV}}x}{\phi'''x}+\frac{1}{48}\frac{\phi'''x}{\phi''x}\right]+\frac{h^2}{2}\left[\frac{1}{192}\frac{\phi^{\mathrm{V}}x}{\phi'''x}-\frac{1}{192}\left(\frac{\phi^{\mathrm{IV}}x}{\phi'''x}\right)^2+\frac{7}{288}\frac{\phi^{\mathrm{IV}}x}{\phi''x}-\frac{1}{48}\left(\frac{\phi'''x}{\phi''x}\right)^2\right]$$
$$+\ldots. \tag{43}$$

or,

$$\theta_2=\frac{1}{4}+h\left(\frac{d\theta_2}{dh}\right)_{h=0}+\frac{h^2}{2}\left(\frac{d^2\theta_2}{dh^2}\right)_{h=0}+\frac{h^3}{2.3}\left(\frac{d^3\theta_2}{dh^3}\right)_{h=0}+\ldots.\frac{h^n}{\underline{|n}}\left(\frac{d^n\theta_2}{dh^n}\right)_{h=0}+\ldots., \tag{44}$$

where

$$\left(\frac{d\theta_2}{dh}\right)_{h=0},\ \left(\frac{d^2\theta_2}{dh^2}\right)_{h=0},\ \ldots.\left(\frac{d^n\theta_2}{dh^n}\right)_{h=0}$$

are expressible in terms of $\phi''x$, $\phi'''x$, ... $\phi^{n+3}x$, as in (40), (41), (42).

If in (5), (6), (7), (8) and (9) we write θ_2 for m, θ_3 for θ, $\phi'''x$ for $\phi'x$, $\phi^{\mathrm{IV}}x$ for $\phi''x$, and so on, then reduce the results, we get

$$\left(\frac{d\theta_3}{dh}\right)_{h=0}=\frac{1}{384}\frac{\phi^{\mathrm{V}}x}{\phi^{\mathrm{IV}}x}+\frac{1}{192}\frac{\phi^{\mathrm{IV}}x}{\phi'''x}+\frac{1}{96}\frac{\phi'''x}{\phi''x}. \tag{45}$$

$$\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}=\frac{1}{1536}\frac{\phi^{\mathrm{VI}}x}{\phi^{\mathrm{IV}}x}-\frac{1}{1536}\left(\frac{\phi^{\mathrm{V}}x}{\phi^{\mathrm{IV}}x}\right)^2+\frac{7}{2304}\frac{\phi^{\mathrm{V}}x}{\phi'''x}-\frac{1}{384}\left(\frac{\phi^{\mathrm{IV}}x}{\phi'''x}\right)^2+\frac{7}{576}\frac{\phi^{\mathrm{IV}}x}{\phi''x}+\frac{1}{1152}\frac{\phi^{\mathrm{V}}x}{\phi^{\mathrm{IV}}x}\frac{\phi'''x}{\phi''x}$$
$$-\frac{1}{96}\left(\frac{\phi'''x}{\phi''x}\right)^2. \tag{46}$$

. . . .

From these values of $\left(\frac{d\theta_3}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}$, ..., we find

$$\theta_3=\frac{1}{8}+h\left(\frac{d\theta_3}{dh}\right)_{h=0}+\frac{h^2}{2}\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}+\ldots.\frac{h^n}{\underline{|n}}\left(\frac{d^n\theta_3}{dh^n}\right)_{h=0}+\ldots. \tag{47}$$

where $\left(\frac{d\theta_3}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}$, ... are expressible in terms of $\phi''x$, $\phi'''x$, ... $\phi^{n+4}x$, as in (45), (46).

In general, it may be shown that

$$\theta_n = \frac{1}{2^n} + h\left(\frac{d\theta_n}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta_n}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta_n}{dh^3}\right)_{h=0} + \dots \frac{h^{n-1}}{\lfloor n-1}\left(\frac{d^{n-1}\theta_n}{dh^{n-1}}\right)_{h=0} + \dots, \tag{48}$$

where $\left(\frac{d\theta_n}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_n}{dh^2}\right)_{h=0}$, ... $\left(\frac{d^{n-1}\theta_n}{dh^{n-1}}\right)_{h=0}$ can be expressed in terms of $\phi''x$, $\phi'''x$, ... $\phi^{2n}x$, as in preceding cases.

Eliminating $\phi'(x+\theta_1 h)$, $\phi''(x+\theta_2 h)$, ... $\phi^{n-1}(x+\theta_{n-1}h)$ from $(a)'$, $(b)'$, ..., there results

$$\phi(x+h) = \phi x + h\phi' x + \theta_1 h^2\phi'' x + \theta_1\theta_2 h^3\phi''' x + \dots \theta_1\theta_2\theta_3\dots\theta_{n-1}h^n\phi^n(x+\theta_n h). \tag{49}$$

Since θ_1 contains h, h^2, h^3, ... as factors, and does not contain h in any other form, and since the same holds true in the expressions giving the values of θ_2, θ_3, ..., it is plain that (49) will contain h as a factor in the regular ascending integral powers, h, h^2, h^3, ..., and will not contain h in any other form, when θ_1, θ_2, ... are replaced by their values in terms of x and h.

From (39), (44), (47), and (48), we readily find

$$\theta_1 h^2\phi'' x = h^2\phi'' x\left[\frac{1}{2} + h\left(\frac{d\theta_1}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0}\right] + \text{terms containing } h^6, h^7, \dots \tag{50}$$

$$\theta_1\theta_2 h^3\phi''' x = h^3\phi''' x\left[\frac{1}{8} + \frac{h}{4}\left(\frac{d\theta_1}{dh}\frac{2\,d\theta_2}{dh}\right)_{h=0} + \frac{h^2}{8}\left(\frac{d^2\theta_1}{dh^2} + \frac{8\,d\theta_1 d\theta_2}{dh dh} + \frac{2\,d^2\theta_2}{dh^2}\right)_{h=0}\right] + \text{terms containing } h^6, h^7, \dots \tag{51}$$

$$\theta_1\theta_2\theta_3 h^4\phi^{\text{IV}} x = h^4\phi^{\text{IV}} x\left[\frac{1}{16} + \frac{h}{32}\left(\frac{d\theta_1}{dh} + \frac{2\,d\theta_2}{dh} + \frac{4\,d\theta_3}{dh}\right)_{h=0}\right] + \text{terms containing } h^6, h^7, \dots \tag{52}$$

$$\theta_1\theta_2\theta_3\theta_4 h^5\phi^{\text{V}} x = h^5\phi^{\text{V}} x\left[\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16}\right] + \text{terms containing } h^6, h^7, \dots \tag{53}$$

In (50), (51), (52), (53), replacing the several diff. co. of θ_1, θ_2, ... by their values in terms of x and h, and substituting the results for $\theta_1 h^2\phi'' x$, $\theta_1\theta_2 h^3\phi''' x$, ... in (49), we get

$$
\begin{aligned}
\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{24}\phi''' x + \frac{h^4}{48}\phi^{\mathrm{IV}} x - \frac{h^4}{48}\frac{(\phi''' x)^2}{\phi'' x} + \frac{11\,h^5}{1920}\phi^{\mathrm{V}} x - \frac{3\,h^5}{192}\frac{\phi''' x}{\phi'' x} + \frac{11\,h^5}{1152}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^3}{8}\phi''' x + \frac{h^4}{192}\phi^{\mathrm{IV}} x + \frac{h^4}{96}\frac{(\phi''' x)^2}{\phi'' x} + \frac{h^5}{768}\phi^{\mathrm{V}} x + \frac{h^5}{192}\frac{\phi''' x}{\phi'' x} - \frac{h^5}{192}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^4}{64}\phi^{\mathrm{IV}} x + \frac{h^4}{96}\frac{(\phi''' x)^2}{\phi'' x} + \frac{h^5}{3072}\phi^{\mathrm{V}} x + \frac{h^5}{2304}\frac{\phi''' x}{\phi'' x} + \frac{h^5}{1152}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^5}{1024}\phi^{\mathrm{V}} x + \frac{7\,h^5}{1152}\frac{\phi''' x}{\phi'' x} - \frac{h^5}{192}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^5}{768}\frac{\phi''' x}{\phi'' x} \\
+ \frac{h^5}{768}\frac{\phi''' x}{\phi'' x} \\
+ \frac{h^5}{768}\frac{\phi''' x}{\phi'' x} \\
- \frac{h^5}{768}\frac{(\phi^{\mathrm{IV}} x)^2}{\phi'' x} + \text{terms containing } h^6, h^7, \ldots, \\
+ \frac{h^5}{1536}\frac{(\phi^{\mathrm{IV}} x)^2}{\phi'' x} \\
+ \frac{h^5}{1536}\frac{(\phi^{\mathrm{IV}} x)^2}{\phi'' x}.
\end{aligned}
$$

Uniting terms,

$$
\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \frac{h^4}{2.3.4}\phi^{\mathrm{IV}} x + \frac{h^5}{2.3.4.5}\phi^{\mathrm{V}} x + \text{terms containing } h^6, h^7, \ldots.
$$

By extending the work, it may be shown that $\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \ldots \frac{h^n}{\lfloor n}\phi^n x$ + such terms in $\theta_1 h^2 \phi'' x$, $\theta_1\theta_2 h^3 \phi''' x$, ... $\theta_1\theta_2$, ... $\theta_{n-1} h^n \phi^n(x+\theta_n h)$ as contain h^{n+1}, h^{n+2}, ..., but no power of h less than h^{n+1}. The sum of all the terms in the right member of the last equation, after the first n, may be denoted by $\frac{h^n}{\lfloor n}\phi^n(x+\theta h)$, giving (18), and the value of θ found as before.